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On a Property of Singular Integrals with Even Positive Kernels

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Let $\mathbf{K}$ be the set of kernels

$$\chi(\rho, u): A_\chi \times [-\pi, \pi] \to \mathbb{R} \qquad (A_\chi \subset \mathbb{R})$$

with:

(i) $\chi(\rho, \cdot) \in C_{2\pi}$, even, non-negative,

(ii) $\int_{-\pi}^{\pi} \chi(\rho, u)\, du = 2\pi$, $\rho \in A_\chi$.

(iii) There exists an accumulation point $\rho_0 \in \mathbb{R} \cup \{-\infty, \infty\}$ of A_χ such that, for every $\delta \in (0, \pi)$,

$$\int_{\delta \leqslant |u| \leqslant \pi} \chi(\rho, u)\, du = o(1), \qquad \rho \to \rho_0.$$

For $f \in L^1_{2\pi}$ (real or complex valued) and $\chi \in \mathbf{K}$, $\rho \in A_\chi$, we define

$$I_\rho(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - u)\, \chi(\rho, u)\, du.$$

Let $f \in C_{2\pi}$ real valued, $\int_{-\pi}^{\pi} f(u)\, du > 0$. Then if $f \geqslant 0$ on $[-\pi, \pi]$ we have

$$|I_\rho(g \cdot f, x)| \leqslant I_\rho(f, x), \qquad x \in [-\pi, \pi], \tag{1}$$

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for every complex valued $g \in L^1_{2\pi}$ with $|g(u)| \leqslant 1$, $u \in \mathbb{R}$. Clearly this bound preserving property is also sufficient for f to be non-negative (for each $\rho \in A_\chi$). In the present paper we deal with the following related conditions:

$$|I_\rho(e^{iu}f(u), x)| \leqslant |I_\rho(f, x)|, \qquad x \in [-\pi, \pi], \quad \text{for all } \rho \in A_\chi. \tag{2}$$

It appears to us that (2), for a large variety of kernels in $\mathbf{K}$, is also sufficient for $f \in C_{2\pi}$, $\int_{-\pi}^{\pi} f(u)\,du > 0$, to be non-negative. Note that this conclusion is trivial if strict inequality is required in (2).

We shall establish this conjecture—occasionally under further restrictions for f—for a number of kernels, among them the kernels of Poisson, Fejér, de la Valleé Poussin and Weierstrass. In the last section we treat the kernel

$$P_2(\rho, u) = \frac{1-\rho^2}{1+\rho^2}\left(\frac{1-\rho^2}{1+\rho^2-2\rho\cos u}\right)^2, \qquad A_{P_2} = (0, 1), \qquad \rho_0 = 1. \tag{3}$$

We have not been able, however, to derive the general condition (if there is any) to be imposed on f or $\chi \in \mathbf{K}$ to make our conjecture valid. It should be pointed out that the desired conclusion fails to hold for suitably chosen f and $\chi \in \mathbf{K}$ with "small" supports.

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Let $b_k(\rho)$ be the Fourier coefficients of $\chi \in \mathbf{K}$,

$$\chi(\rho, u) \sim \sum_{k=-\infty}^{\infty} b_k(\rho)\, e^{iku} \tag{4}$$

such that $b_0(\rho) \equiv 1$, $b_k(\rho) = b_{-k}(\rho) \in \mathbb{R}$ for $k \in \mathbb{N}$. For $j \in \mathbb{Z}$ we define

$$\Delta_2 b_j(\rho) = b_j(\rho) - 2b_{j+1}(\rho) + b_{j+2}(\rho)$$

and for $m > 1$

$$\Delta_{2m} b_j = \Delta_{2m-2} b_j - 2\Delta_{2m-2} b_{j+1} + \Delta_{2m-2} b_{j+2}.$$

THEOREM 1. *Let $\chi \in \mathbf{K}$ such that*

(i) $\chi(\rho, \cdot) \in C^2_{2\pi}$, $\rho \in A_\chi$,

(ii) $\sum_{k=1}^{\infty} |\Delta_2 b_{k-1}(\rho)| = O((1 - b_1(\rho)))$, $\rho \to \rho_0$,

(iii) $\Delta_2 b_{k-1}(\rho) = o(1 - b_1(\rho))$, $\rho \to \rho_0$, $k \in \mathbb{N}$.

Then if (2) *holds for $f \in C_{2\pi}$ with $\int_{-\pi}^{\pi} f(u)\,du > 0$ we have $f \geqslant 0$ in $[-\pi, \pi]$.*

Proof. Without loss of generality we assume $I_{\rho_n}(f, x_n) = 0$ for a sequence of pairwise disjoint $\rho_n \in A_\chi$, $n \in \mathbb{N}$, with $\rho_n \to \rho_0$ and corresponding numbers x_n (otherwise the conclusion is trivial for the approximate identity χ). Hence $I_{\rho_n}(e^{iu}f(u), x_n) = 0$, $n \in \mathbb{N}$. If

$$f(u) \sim \sum_{k=-\infty}^{\infty} a_k e^{iku}, \qquad a_k = \overline{a_{-k}},$$

then since $\chi \in C_{2\pi}^2$

$$\operatorname{Re} I_{\rho_n}(f, x_n) = a_0 + 2 \sum_{k=1}^{\infty} b_k(\rho_n) \operatorname{Re}(a_k e^{ikx_n}) = 0$$

and

$$\begin{aligned}\operatorname{Re}&[e^{-ix_n} I_{\rho_n}(e^{iu}f(u), x_n)]\\ &= a_0 b_1(\rho_n) + \sum_{k=1}^{\infty} (b_{k-1}(\rho_n) + b_{k+1}(\rho_n)) \operatorname{Re}(a_k e^{ikx_n}) = 0.\end{aligned}$$

This implies

$$a_0(b_1(\rho_n) - 1) + \sum_{k=1}^{\infty} (\Delta_2 b_{k-1}(\rho_n)) \operatorname{Re}(a_k e^{ikx_n}) = 0$$

for $n \in \mathbb{N}$. Using (ii), (iii) of the assumption and the fact $a_k \to 0$, $k \to \infty$, we obtain

$$\sum_{k=1}^{\infty} \frac{\Delta_2 b_{k-1}(\rho_n)}{1 - b_1(\rho_n)} \cdot \operatorname{Re}(a_k e^{ikx_n}) = o(1), \qquad n \to \infty,$$

which gives the contradiction

$$0 = a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\, du > 0.$$

For Fejér's kernel

$$F(n, u) = \frac{1}{n+1} \left(\frac{\sin((n+1)(u/2))}{\sin(u/2)}\right)^2 = 1 + 2 \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \cos ku,$$

$A_F = \mathbb{N}$, $\rho_0 = \infty$, and for Poisson's kernel

$$P(\rho, u) = \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos u} = 1 + 2 \sum_{k=1}^{\infty} \rho^k \cos ku,$$

$A_p = (0, 1)$, $\rho_0 = 1$, the assumptions of Theorem 1 are fulfilled and our conjecture is therefore proved in these cases.

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For Korovkin-type kernels $\chi \in \mathbf{K}$, i.e., those with

$$\frac{1 - b_2(\rho)}{1 - b_1(\rho)} = 4 + o(1), \qquad \rho \to \rho_0,$$

we have for $k \in \mathbb{N}$

$$\frac{\Delta_2 b_{k-1}(\rho)}{b_1(\rho) - 1} = 2 + o(1), \qquad \rho \to \rho_0,$$

and Theorem 1 is not applicable. For these kernels we do not have a general method to establish our conjecture. We can prove it, however, for those Korovkin-type kernels which admit an infinite trigonometric asymptotic expansion of $I_\rho(f, x)$ and real-analytic f.

Let $\mathbf{K}'$ denote the class of those $\chi \in \mathbf{K}$ for which

$$\Delta_{2m+2}(b_{-m-1}(\rho)) = o(\Delta_{2m}(b_{-m}(\rho))), \qquad \rho \to \rho_0$$

holds for $m \in \mathbb{N}$. We make use of the following Lemma which is easily deduced along the lines in [3, pp. 73–75] and can also be extracted from the results in [2].

LEMMA. *Let $\chi \in \mathbf{K}'$ and $f \in C_{2\pi}^{2m}$, $f^{(j)}(x_0) = 0$, $j = 0, 1, \dots, 2m-2$. Then for $\rho \to \rho_0$*

$$I_\rho(f, x_0) = \frac{(-1)^m}{(2m)!} f^{(2m)}(x_0)\, \Delta_{2m}(b_{-m}(\rho)) + o(\Delta_{2m}(b_{-m}(\rho))). \tag{5}$$

THEOREM 2. *Let $\chi \in \mathbf{K}'$ and $f \in C_{2\pi}^{\infty}$ real analytic with $\int_{-\pi}^{\pi} f(u)\, du > 0$. Then the conditions* (2) *imply $f \geqslant 0$ on $[-\pi, \pi]$.*

Proof. We show that every zero of f is of even multiplicity which implies the assertion. In fact, if $f^{(j)}(x_0) = 0$ for $0 \leqslant j \leqslant 2m-2$ and $f^{(2m-1)}(x_0) \neq 0$, a combination of (5) and (2) gives for $\rho \to \rho_0$

$$|2mif^{(2m-1)}(x_0) + f^{(2m)}(x_0) + o(1)| \leqslant |f^{(2m)}(x_0) + o(1)|$$

which implies the contradiction $f^{(2m-1)}(x_0) = 0$.

It follows from the results in [2] that the de la Valleé Poussin kernel

$$V(n, u) = \frac{(n!)^2}{(2n)!}\left(2\cos\frac{u}{2}\right)^{2n}, \qquad n \in \mathbb{N},$$

is in $\mathbf{K}'$. The same is true for the Weierstrass kernel

$$W(\rho, u) = 1 + 2\sum_{k=1}^{\infty} e^{-\rho k^2}\cos ku, \qquad \rho \to 0+ ,$$

as an iterated application of l'Hospital's rule to

$$\frac{\Delta_{2m-2}b_{-m-1}(\rho)}{\Delta_{2m}b_{-m}(\rho)}, \qquad \rho \to 0+ , m \in \mathbb{N}$$

shows.

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In this concluding section we establish our conjecture for the kernel

$$P_2(\rho, u) = \frac{1-\rho^2}{1+\rho^2}\left(\frac{1-\rho^2}{1+\rho^2-2\rho\cos u}\right)^2$$

$$= 1 + 2\sum_{k=1}^{\infty}\left(\frac{1-\rho^2}{1+\rho^2}k + 1\right)\rho^k\cos ku,$$

$A_{p_2} = (0, 1)$, $\rho_0 = 1$, which does not belong to $\mathbf{K}'$ but is of Korovkin-type. It is known [1, pp. 205–210] that for real or complex valued functions $h \in C_{2\pi}$ we have

$$I_\rho(h, x) = \frac{1-|z|^2}{1+|z|^2}(zH_z + \bar{z}H_{\bar{z}}) + H, \qquad z = \rho e^{ix},$$

where H denotes the harmonic function in $|z| < 1$ with the continuous boundary values h. Now if F denotes the analytic function in $|z| < 1$ with $F(0) \in \mathbb{R}$ and $\operatorname{Re} F$ harmonic with the boundary values f it is clear that the harmonic function H with the boundary values $e^{iu}f(u)$ is given by

$$2H(z) = zF(z) + \overline{\left(\frac{F(z)}{z}\right)} + \left(z - \frac{1}{\bar{z}}\right)F(0)$$

such that condition (2) reads in this particular case

$$|(1-|z|^2)(z^2F'+\overline{F'})+2z(F+\bar{F})| \leqslant |(1-|z|^2)(zF'+\overline{zF'})+(1+|z|^2)(F+\bar{F})|$$

for $|z|<1$. A rearrangement leads to the equivalent inequality

$$(1-|z|^2)\,|F'(z)| \leqslant 2\,|\mathrm{Re}\, F(z)|, \qquad |z|<1. \tag{6}$$

Equation (6) is well known to hold for analytic F with $F(0)>0$, $\mathrm{Re}\, F(z)>0$ in $|z|<1$. It remains to show the sufficiency of (6) for $\mathrm{Re}\, F(z)\geqslant 0$ if F is analytic in $|z|<1$, $F(0)>0$. Assume $\mathrm{Re}\, F(z_0)=0$ for a certain $|z_0|<1$. Then it follows from (6)

$$\lim_{z\to z_0}(1-|z|^2)\frac{|F'(z)|}{|F(z)-F(z_0)|}\leqslant 2$$

which is impossible.

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